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GROUP OF RATIONAL NUMBERS AS A SUM OF SQUARES (Q^{+2})

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ABSTRACT:

Let $\mathbf{Q}^{+2}$ denotes the set of non-negative rational numbers which can be express as sum of square of two rational. In this paper we will prove that $\mathbf{Q}^{+2}$ is a group over multiplication.

Definition:

Let Q denotes the set of rational numbers. Then $\mathbf{Q}^{+2}$ denotes the set of non-negative rational numbers. i.e $\mathbf{Q}^{+2} = \{n \in Q | \ n = a^2 + b^2 \text{ for some } a, b \in Q\}$.

Since $3^2 + 4^2 = 5^2 \rightarrow 5 \in \mathbf{Q}^{+2} \rightarrow \mathbf{Q}^{+2} \neq \phi$

Theorem No. 01:

Let $G = (\mathbf{Q}^{+}, \ \cdot \ )$, be a set then G is abelian group.

Proof:

I) Closure property:

Let $n, m \in \mathbf{Q}^{+2}$

$\therefore \exists a, b, c, d \in Q$ such that $n = a^2 + b^2$ & $m = c^2 + d^2$

Consider,

$$\begin{aligned} n \cdot m = (a^2 + b^2)(c^2 + d^2) &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2 \\ &= (ac)^2 - 2(ac)(bd) + (bd)^2 + (ad)^2 + 2(ad)(bc) + (bc)^2 \\ &= (ac - bd)^2 + (ad + bc)^2 \in Q^{+2} \end{aligned}$$

$\therefore$ Q^{+2} is closure under multiplication, $\cdot$

II) Associativity:

Multiplication for rational numbers is always associative.

Also $Q^{+2} \subseteq Q$.

Therefore, Q^{+2} is Assocoative under multiplication.

III) Existence of Identity:

We have for all $n \in Q^{+2}, n \cdot 1 = n = 1 \cdot n$

Also $1 = 0^2 + 1^2, \therefore 1 \in Q^{+2}$

$\therefore$ 1 is identity in Q^{+2}.

IV) Existence of inverse:

Let $n(\neq 0) \in Q^{+2}$,

$\exists\ a, b \in Q$ such that $n = a^2 + b^2$

Consider, $m = \frac{1}{n} = \frac{1}{a^2 + b^2} = \frac{a^2 + b^2}{(a^2 + b^2)^2} = (\frac{a}{a^2 + b^2})^2 + (\frac{b}{a^2 + b^2})^2 \in Q^{+2}$

$\therefore m \cdot n = 1 = n \cdot m$

=$\therefore \exists$ multiplicative inverse for every non zero element of Q^{+2}.

Hence $G = (\mathbf{Q}^+, \cdot)$, is group.

Also, multiplication in rational numbers is commutative.

Therefore,

$G = (\mathbf{Q}^+, \cdot)$, is abelian group.

Corollary No. 01:

Let $n, m \in Q^{+2}$, then $\frac{n}{m} \in Q^{+2}$

Proof:

Since $n, m \in Q^{+2}$,

$\therefore \exists a, b, c, d \in Q$ such that $n = a^2 + b^2$ & $m = c^2 + d^2$

$$\therefore \frac{n}{m} = \frac{a^2 + b^2}{c^2 + d^2} = \frac{(a^2 + b^2)(c^2 + d^2)}{(c^2 + d^2)(c^2 + d^2)} = \frac{(ac - bd)^2 + (ad + bc)^2}{(c^2 + d^2)^2}$$
$$= (\frac{ac - bd}{c^2 + d^2})^2 + (\frac{ad + bc}{c^2 + d^2})^2 \in Q^{+2}$$

Corollary No. 02:

Let $n \in Q^{+2}$ *and* $m \notin Q^{+2}$, then $n \cdot m \notin Q^{+2}$

Proof:

$Since$ $n \in Q^{+2}$,

$\therefore \exists a, b \in Q$ such that $n = a^2 + b^2$

Suppose if possible that, $n \cdot m \in Q^{+2}$.

$\therefore \exists$ c, d $\in Q$ such that $n \cdot m = c^2 + d^2$

$$\therefore m = \frac{c^2 + d^2}{a^2 + b^2} \in Q^{+2}, \qquad by\ corollary\ no.\ 01$$

$which\ is\ a\ contradiction\ to\ the\ given\ that\ m \notin Q^{+2}$.

Hence proved.

Examples:

1)Check whether 13325 can be express as a sum of two squares.

Consider,

$13325 = 25 \times 533 = 25 \times 13 \times 41 = (3^2 + 4^2)(2^2 + 3^2)(4^2 + 5^2)$

$By\ Theorem\ No.\ 01$,

$13325 \in Q^{+2}$,

$\therefore \exists\ a, b \in Q\ such\ that\ 13325 = a^2 + b^2$

In fact,

$$\begin{aligned} 13325 &= ((6 + 12)^2 + (9 - 8)^2)(4^2 + 5^2) = (18^2 + 1^2)(4^2 + 5^2) \\ &= (72 - 5)^2 + (90 + 4)^2 = 67^2 + 94^2 \end{aligned}$$

REFERENCE

Sibner, Robert J. "Fermat Theorems--Simple Proofs." arXiv preprint arXiv:2109.10220 (2021).